Weil–Moore anima learning seminar

SLMath
Motivic homotopy theory program

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Introduction

Let K be a number field with ring of integers $\mathcal{O}_K$. Mazur and Mumford 'knots and primes' analogy [Maz64; Mor12], we should think of $\mathrm{Spec}(\mathcal{O}_K)$ as the interior of a compact 3-manifold with boundary. More precisely, for the analogy we should work with the ring of S-integers $\mathcal{O}_{K,S}$, whose étale fundamental group is the Galois group $\mathrm{G}_{K,S}$ of the maximal field extension of K that is unramified outside of S. The reason is the following. Let p be a prime, S a finite set of places of K containing all archimedean places and all places above p, and let M be a finite $\mathrm{G}_{K,S}$-module of p-power order. *Poitou–Tate duality* [Mil06; Tat63] provides a 3-dimensional 'Poincaré duality'

$$\mathrm{R\Gamma}(\mathrm{BG}_{K,S}; M)^\vee \simeq \widetilde{\mathrm{R\Gamma}}_{\mathrm{c}}(\mathrm{BG}_{K,S}; M^\vee(1)[3]) \,.$$

In this analogy, the set S is meant to parametrize the boundary components of the 3-manifold, and the Tate twist should be interpreted as giving the orientation local system.

Some explanation of the above terms is in order: the left-hand side denotes the Pontryagin dual of continuous cohomology, but the right-hand side is not simply a form of compactly supported cohomology. Rather, it is defined in an ad hoc manner as the fiber

$$\widetilde{\mathrm{R\Gamma}}_{\mathrm{c}}(\mathrm{BG}_{K,S}; N) \coloneqq \mathrm{fib}\left(\mathrm{R\Gamma}(\mathrm{BG}_{K,S}; N) \to \bigoplus_{v\in S} \widetilde{\mathrm{R\Gamma}}(\mathrm{BG}_{K_v}; N)\right) .$$

In the last terms, we also have to modify usual cohomology: $\widetilde{\mathrm{R\Gamma}}$ denotes continuous group cohomology if the place v is nonarchimedian, and Tate cohomology if the place v is archimedian (so G_{K_v} is a finite group).

The goal of this seminar is to study Dustin Clausen's recent paper on *Weil–Moore anima* [Cla26e], which improves this analogy. Clausen's solution is to replace BG_K by an anima X_K equipped with a map $\mathrm{X}_K \to \mathrm{BG}_K$. Clausen also constructs variants $\mathrm{X}_{K,S} \to \mathrm{BG}_{K,S}$ with restricted ramification, whose cohomology satisfies a natural 'Poincaré duality' theorem recovering Poitou–Tate duality.

The first (well-known) observation about how to improve the situation is that we should replace the Galois group of K by the *Weil group* of K. This is a locally compact Hausdorff group W_K that refines G_K in the sense that there is a surjective homomorphism of topological groups

$$\mathrm{W}_K \twoheadrightarrow \mathrm{G}_K$$

with connected kernel. It was introduced by Weil [Wei51] to unify the theories of Artin L-functions and Hecke L-functions. The Weil group also has better cohomological properties, and fixes some issues with Galois cohomology in degree 2. Unfortunately, the higher cohomology still does not behave as desired. Clausen's work corrects this by refining BW_K to a connected condensed[1] anima X_K whose fundamental group is the Weil group W_K, and whose higher-degree cohomology has the desired properties. This is analogous to the behavior of Moore anima, hence X_K is called the *Weil–Moore anima* of K.

Clausen also provides a number of interesting applications of Weil–Moore anima, and variants and reformulations of older results in terms of Weil–Moore anima. Beyond the construction of Weil–Moore anima, one of our primary aims is to understand these applications.

Resources

In addition to Clausen's paper [Cla26e], Clausen gave a series of four lectures at IHÉS that may be useful to watch [Cla26a; Cla26b; Cla26c; Cla26d].

Pre-reading

Before the seminar, please skim the introduction [Cla26e, §1]. Also read the background on condensed mathematics [Cla26e, §2]. Undertstand the definition of condensed anima, ignoring set-theoretic issues. Understand how compact Hausdorff spaces sit inside condensed anima. Read the the cohomology comparison of Theorem 2.2 and the Lie group calculation of Lemma 2.4 from bar resolutions. Read about derived Pontryagin duality in its ordinary and equivariant forms. Understand why classifying anima of compact Hausdorff groups are closed under cofiltered limits (Lemma 2.14). The results of [Cla26e, §2] are mostly technical/generalizations of more classical results and are used in places scattered throughout the paper.

Talks

(1) **Overview** [Cla26e, §1]

Explain the construction of Weil groups from cohomological class field theory. State Theorem 1.2 and isolate the unwanted high-degree cohomology of the archimedean Weil groups. For the variants over $\mathbf{C}$ and $\mathbf{R}$, work out the guiding replacements of $\mathbf{CP}^\infty$ by S^2 and of $\mathbf{RP}^\infty$ by $\mathbf{RP}^2$. In the real case, explain the relationship to the twistor line. Sketch an outline for the seminar.

(2) **"Almost compact" condensed anima** [Cla26e, §3]

Define *almost compact* groups and prove their structure and permanence properties. Starting with compact fundamental group, derive Proposition 3.6 for the cohomology of condensed anima with almost compact fundamental group. Deduce the $\mathbf{R}/\mathbf{Z}$ calculation in Corollary 3.7, introduce X°, and explain how Lemma 3.11 classifies the next Postnikov stage. State Theorem 3.14, then construct the underlying ordinary anima and profinite completion. For many proofs, just give a sketch or explain the key idea behind the proof. For example, the more classical fact that every compact Hausdorff group is the limit of a cofiltered diagram of Lie groups.

[1] Since the Weil group is a topological group (which generally isn't even profinite), one needs a setting in which homotopy groups are 'topological' groups. Condensed mathematics provides such a setting.

(3) **Moore anima & obstruction theory** [Cla26e, §4]

Define *Moore anima* and *n-stage Moore anima*. Analyze the obstructions to the existence of 1-stage Moore anima as well as to lifting from n-stage to $(n+1)$-stage Moore anima. Prove that for almost compact groups with abelian connected component of the identity, the 1-stage obstruction vanishes (Proposition 4.11). Introduce the cohomology object c_G. Prove Theorem 4.15, identifying the anima of lifts with nullhomotopies of a canonical map to $c_G[n+3]$. Deduce the existence and uniqueness criterion for Moore anima (Corollary 4.16).

(4) **Class formations, transfers, and the Tate–Nakayama lemma** [Cla26e, §5]

Recall classical class formations (see [AT09]), then define a *refined class formation* as a two-term cosheaf F with an augmentation $\mathrm{H}_0(F) \to \mathbf{Z}$. Explain sheaves and cosheaves on Galois categories and construct transfer maps via spans. State the Tate–Nakayama lemma.

Explain Theorem 5.10: a refined class formation is a class formation together with a map $\kappa : \frac{1}{\#\mathcal{C}}\mathbf{Z} \to \mathrm{R}\Gamma(\mathcal{C}; A[2])$ inducing an isomorphism

$$\left(\frac{1}{\#\mathcal{C}}\mathbf{Z}\right)/\mathbf{Z} \simeq \mathrm{H}^2(\mathcal{C}; A)\,.$$

Compute the homotopy of the anima of refinements (Corollary 5.11). Work through some examples. Define *condensed refined class formations* and explain Lemma 5.15. Define morphisms of refined class formations and state Corollary 5.20.

(5) **Refined class formations as Weil groupoids** [Cla26e, §6, Definition 6.1–Corollary 6.19]

Introduce finite étale maps of condensed anima and explain how Galois categories embed into condensed anima. Explain how to view a condensed refined class formation on a Galois category $\mathcal{C}$ as a functor $\mathcal{C}_{\mathrm{conn}} \to \mathrm{Cond}(\mathbf{An})$ from the connected objects of $\mathcal{C}$ to condensed anima (Proposition 6.8). Use this to define the *Weil* condensed groupoid $\mathcal{W}_{\mathcal{C}}$. Explain the 'finite approximation' result (Lemma 6.13) for $\mathcal{W}_{\mathcal{C}}$. Explain why, under mild hypotheses, $\mathcal{W}_{\mathcal{C}}$ is connected (Proposition 6.17). Explain why the datum of the Weil groupoid $\mathcal{W}_{\mathcal{C}}$ is equivalent to a condensed refined class formation (Corollary 6.19).

(6) **The existence theorem for Weil–Moore anima** [Cla26e, §§6–7, Definition 6.20–Remark 7.7]

Define condensed refined class formations of $\mathbf{R}$-*type*, $\mathbf{Z}$-*type*, and *compact-type*. Explain the strict cohomological dimension description of $\mathbf{Z}$-type class formations (Proposition 6.21). Define Weil–Moore anima for condensed refined class formations (Definition 7.2). Prove that the dual of their $\mathbf{R}/\mathbf{Z}$-cohomology complex recovers the refined class formation (Lemma 7.3). Sketch the Ext-vanishing argument of Proposition 7.4. Deduce the main result, Theorem 7.6.

(7) **Maps, archimedean models, and π_2** [Cla26e, §7, Example 7.8–Remark 7.13]

Describe the Weil–Moore anima $\mathrm{X}_{\mathbf{C}}$ and $\mathrm{X}_{\mathbf{R}}$ using S^2, $\mathbf{RP}^2$, quaternions, and the twistor line. Identify the obstruction to lifting maps of condensed refined class formations (Proposition 7.9). Compute that for a condensed refined class formation $\mathcal{C}$ with Weil–Moore anima $\mathrm{X}_{\mathcal{C}}$, we have

$$\pi_2(\mathrm{X}_{\mathcal{C}}) \simeq (\mathrm{Sym}^2_{\mathbf{Z}}\,\Lambda)^{\vee}\,,$$

where Λ is a torsionfree discrete $\mathbf{Z}$-module naturally associated to the class formation (Lemma 7.11). Explain Remark 7.13 on the independence of choices after profinite completion.

(8) **Comparison with Galois cohomology** [Cla26e, §8, Theorem 8.1–Remark 8.19]

Start proving the Weil–Moore analogs of some theorems of Tate. Establish the p-finiteness criterion of Proposition 8.5. Explain the 'Euler characteristic' result in Theorem 8.9. Prove Lemma 8.17 comparing Galois cohomology with cohomology of the Weil–Moore anima. Sketch Theorem 8.19.

(9) **Poitou–Tate duality** [Cla26e, §8, Remark 8.20–Theorem 8.41]

State the technical background needed to prove Poitou–Tate duality for Weil–Moore anima. Construct the local fundamental class and prove local Tate duality (Theorem 8.33). Prove global Poitou–Tate duality (Theorem 8.35). Explain Theorem 8.41, the Weil–Moore version of Poitou's general duality theorem for class formations [Poi61].

References

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