On the dynamical properties of geodesic flow

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1 Introduction

The object of dynamical systems is to study the evolution and behavior of a system over time, where 'system' refers to one of the following:

- A measure space $(X, \mathcal{B}, \mu)$ with a family of measure-preserving maps $T_n : X \to X$, i.e. for all $A \subseteq X$ measurable, $\mu(T^{-1}(A)) = \mu(A)$, such that $T_n \circ T_m = T_{n+m}$.
- A topological space (Y, τ) with a family of continuous maps $f_n : Y \to Y$ such that $f_n \circ f_m = f_{n+m}$.

We have been deliberately vague about the time–we can either take $n, m \in \mathbb{N}$(or $\mathbb{Z}$) or $n, m \in \mathbb{R}_{\geq 0}$(or $\mathbb{R}$), and we refer to these situations as *discrete (resp. continuous) time* systems (and in the latter case, we write t instead of n). If our time variable includes negative numbers, we say that the system is *reversible*.

We study these systems up to *conjugacy:*

Definition 1. Let $(X_i, \mathcal{B}_i, \mu_i), i = 1, 2$ be two probability spaces and let $T_i : (X_i, \mathcal{B}_i, \mu_i) \to (X_i, \mathcal{B}_i, \mu_i)$ be measure-preserving maps. Write $\tilde{T}_i^{-1} : \mathcal{B}_i \to \mathcal{B}_i$ for the induced maps on σ-algebras. Then we say that T_1 and T_2 are *conjugate* if there exists a measure algebra isomorphism $\Phi : (\mathcal{B}_2, \mu_2) \to (\mathcal{B}_1, \mu_1)$ such that $\Phi\tilde{T}_2^{-1} = \tilde{T}_1^{-1}\Phi$.

Some questions that naturally arise are:

- How do we describe and quantify the qualitative behavior of T?
- Can we find a computable invariant h of dynamical systems, i.e. such that if T, T' are conjugate, then $h(T) = h(T')$? If such an h exists, is it a complete invariant?

The goal of this paper is to give partial answers to the above questions, and then explore those answers in the context of a particular system, the geodesic flow on a closed manifold of negative sectional curvature.

We start with some simple examples which will (a) illustrate *how* our definitions work/inform our intuition and (b) provide computational tools and scaffolding by allowing us to reduce the study of some more complex systems to these examples.

Example 2 (Two-sided shift). Let $k \in \mathbb{Z}_{\geq 0}$ and consider $X := \prod_{\mathbb{Z}}\{0, 1, \ldots, k\}$. Given a finite sequence $(a_1, \ldots, a_n) \in \{0, 1, \ldots, k-1\}$, it turns out that specifying $\mu(\{(x_\ell) \mid x_{q+1} = a_1, \ldots \quad x_{q+n} = a_n\}) = \frac{1}{k^n}$ gives a unique, well-defined measure on X, and the shift operator $T : X \xrightarrow{\sim} X$, $T((x_k)_{k\in\mathbb{Z}})_\ell = x_{\ell-1}$ is measure-preserving.

The previous example can be generalized as follows.

Example 3 (Markov shift). Let $P = (p_{ij})_{0\leq i,j\leq k-1}$ be a stochastic matrix, i.e. a matrix with nonnegative entries such that all the columns sum to 1, and let $\vec{p} = (p_0, \ldots, p_{k-1})$ be a probability vector satisfying $\sum_{j=0}^{k-1} p_{ji}p_i = p_j$. Then we can modify the previous definition such that $\mu(\{(x_\ell) \mid x_q = a_0 \quad x_{q+1} = a_1 \quad x_{q+n} = a_n\}) = p_{a_0}p_{a_1a_0}\cdots p_{a_na_{n-1}}$, and X with the shift operator T (but a different *measure*) is referred to as a *topological Markov chain* X_P or the *two-sided* $(P, \vec{p})$*-shift.* If A is the $k \times k$ adjacency matrix

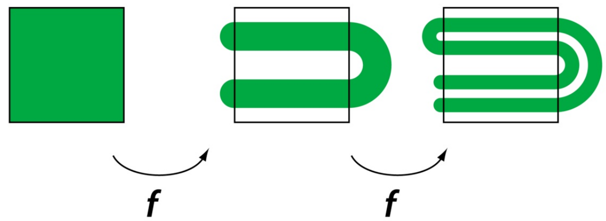

Figure 1: Two iterates of the horseshoe map (image taken from [20]).

of a(n unweighted, finite) graph, we associate to A a stochastic matrix P by dividing each column by the sum of its elements and we also write X_A for the topological Markov chain associated to $(P, (1/k, \dots, 1/k))$.

Remark 4. Note that the condition on the measure in the previous example says that 'inadmissible sequences have measure zero.' From the topological point of view, an equivalent construction would be to simply discard the inadmissible sequences/restrict to the subspace of 'admissible' shifts, i.e. the subspace on those $(x_n)_{n\in\mathbb{Z}}$ such that $p_{x_n x_{n-1}} \neq 0$ for all n. Of course, the topological point of view also discards information recorded by the measure.

Example 5 (Horseshoe map). Another example of a discrete time system is the horseshoe map, which 'stretches and folds' (see Figure 1).

Outline First, we introduce ergodicity, consider whether a few examples are ergodic, and state the Birkhoff Ergodic Theorem. Next, we introduce both measure-theoretic and topological entropy, give computational tools for computing each, and relate the various definitions of entropy. We briefly introduce geodesic flow on the unit tangent bundle of a Riemannian manifold, then prove that the rate of volume growth in the universal cover of a Riemannian manifold bounds the topological entropy of its geodesic flow from below. Finally, we state the ergodicity of geodesic flow on a closed, Riemannian manifold of negative sectional curvature and sketch the proof.

Conventions. We assume that all measures are σ-additive. When a measure or metric is fixed, we drop it from the notation. In the following, $T = T_1$ typically denotes a map corresponding to a discrete time dynamical system, and we typically write $\psi = (\psi_t)_{t\in\mathbb{R}}$ for continuous time systems, a.k.a. flows. Often, a continuous system ψ is said to have a property defined for discrete time systems if its time 1 map ψ_1 has said property. The flows we consider here are reversible, but we do not assume a discrete system T is reversible unless explicitly indicated. In the following, most definitions have slightly different but completely analogous versions for discrete time systems vs. continuous time systems, e.g. by simply replacing T by ϕ and $\forall n \in \mathbb{Z}(\mathbb{N})$ by $\forall t \in \mathbb{R}_{(\geq 0)}$. We state all definitions for the discrete time case and give an indication for how some may be generalized for continuous time, but others are left to the reader. Similarly, we state all definitions in their measure-theoretic versions and leave the reader to fill in the details occasionally when a topological version is immediate (e.g. *topological conjugacy* as a counterpart to Definition 1). We assume background in differential geometry at the level of [13] and some knowledge of measure theory.

The bulk of §2 and §3 can be found in Chapters 4 and 7 of [19], unless otherwise noted.

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2 Ergodicity

Let $(X, \mathcal{B}, \mu)$ be a probability space throughout this section, and let $T : (X, \mathcal{B}, \mu) \to (X, \mathcal{B}, \mu)$ be a measure-preserving transformation. Note that if there was a measurable $B \subseteq X$ such that $T^{-1}(B) = B$ and $T^{-1}(B^c) = B^c$, then we could study the restriction of T to B and its complement, respectively, so T is 'decomposable.' However, if (WLOG) B^c had measure zero, then we should consider T and $T|_B$ to be equivalent. This motivates our definition for 'semisimplicity' of a dynamical system.

Definition 6. We say T is *ergodic* if $T^{-1}B = B \implies \mu(B) = 0$ or $\mu(B) = 1$.

If $(\psi_t)_{t\in\mathbb{R}}$ is a flow, then we say that (ψ_t) is *ergodic* if $\psi_t^{-1}B = B$ for all t implies $\mu(B) = 0$ or $\mu(B) = 1$.

Proposition 7. *Let $(X, \mathcal{B}, \mu)$ be a probability space and $T : X \to X$ a measure-preserving transformation. Then the following are equivalent*

1. *T is ergodic.*
2. *The only measurable subsets $B \subseteq X$ with $\mu(B \setminus T^{-1}B \cup T^{-1}B \setminus B) = 0$ are those B with $\mu(B) = 0$ or $\mu(B) = 1$.*
3. *For all measurable A with $\mu(A) > 0$, we have that $\mu\left(\bigcup_{n\geq 1} T^{-n}A\right) = 1$.*
4. *For every A, B such that $\mu(A), \mu(B) > 0$, there exists $n > 0$ such that $\mu(A \cap T^{-n}B) > 0$.*

Note that such T also acts on measurable functions $f : X \to \mathbb{R}$ via pullback: $T^*(f) := f \circ T$, and we can also characterize ergodicity of T in terms of how it acts on functions.

Proposition 8. *Let $(X, \mathcal{B}, \mu)$, $T : X \to X$ be as above and let $f : X \to \mathbb{R}$. Then TFAE:*

1. *T is ergodic.*
2. *For any measurable $f : X \to \mathbb{R}$ such that $T^*f(x) = f(x)$ for all $x \in X$, f is constant a.e.*
3. *For any measurable $f : X \to \mathbb{R}$ such that $T^*f(x) = f(x)$ a.e., f is constant a.e.*

Remark 9. We call functions satisfying condition 2 of Proposition 8 *T-invariant*, and functions satisfying condition 3 *strictly T-invariant*.

We will use the following proposition in the proof of Theorem 66.

Proposition 10. *Let T (resp. (ϕ_t)) be a measurable function (resp. flow) on a measure space $(X, \mathcal{B}, \mu)$, and let $f : X \to \mathbb{R}$ be a T-(resp. ϕ-)invariant function. Then there is a strictly T-(resp. ϕ-)invariant measurable function $\tilde{f}$ such that $f = \tilde{f}$ a.e.*

Recall: any locally compact topological group G has a unique measure μ which is invariant with respect to left multiplication, which is referred to as the *Haar measure*. If G is compact, then $\mu(G) < \infty$.

Example 11. Let $X = S^1$ considered as a subset of the complex plane, and let μ be the Haar measure on S^1. Then for any angle $\theta \in S^1$, the rotation $T : S^1 \to S^1, z \mapsto \theta z$ is ergodic if and only if θ is *not* a root of unity.

The previous example can be generalized in the following way:

Proposition 12. *Let G be any compact topological group and let $T_a : G \to G$ be left multiplication by some element $a \in G$. Then T_a is ergodic iff $\{a^n \mid n \in \mathbb{Z}\}$ is dense in G.*

In particular, if T_a is ergodic, then G is abelian.

Example 13. Let T be the topological Markov shift. Then T is ergodic if and only if the matrix P is irreducible, i.e. if, for all i, j there exists $n > 0$ such that $(P^n)_{ij} > 0$. Note that if P comes from the adjacency matrix of a graph, this is equivalent to asking that the graph be connected.

Let T (resp. (ψ_t)) be an ergodic transformation (resp. flow). From the equivalent conditions in Proposition 7, one interpretation is that a set A of positive measure visits, or leaves a 'footprint' (under T, resp. ψ iteration) everywhere in X. The following theorem makes this precise, and shows that the footprint is *uniformly distributed* in some sense:

Theorem 14 (Birkhoff)**.** *Let $(X, \mathcal{B}, \mu)$ be a probability space and let $f \in L^1(X, \mathcal{B}, \mu)$.*

1. Let T be a measure-preserving map $T : X \to X$. Then

$$\frac{1}{n}\sum_{i=0}^{n-1} f(T^i(x))$$

converges a.e. to a function $f^* \in L^1(X, \mathcal{B}, \mu)$. Moreover, f^* is T-invariant, and $\int f^* d\mu = \int f d\mu$.

2. Let (ψ_t) be a measure-preserving flow $\psi_t : X \to X$. Then

$$f^+(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^T f(\psi_t x)dt \qquad f^-(x) = \lim_{T\to\infty} \frac{1}{T}\int_0^{-T} f(\psi_t x)dt$$

exist and are equal a.e. Moreover, $f^\pm$ are μ-integrable and ψ-invariant, and $\int f d\mu = \int f^\pm d\mu$.

Remark 15. It follows from the above that if T is ergodic, then f^* is constant a.e., so

$$\int_X f d\mu = \lim_{n\to\infty} \sum_{i=0}^{n-1} f(T^i(x)) \ ,$$

hence the Birkhoff ergodic theorem is often summarized as 'time average equals space average.'

3 Entropy

We define measure-theoretic entropy and topological entropy, motivate the definition via information theory, and state a variational principle relating the two. We start by introducing entropy of a discrete time dynamical system, then explaining how it can be generalized to a continuous time dynamical system. The reason for this is two-fold: because the definition is easier to state for the discrete time case, and also because we will see examples where we can estimate the entropy of a continuous flow via the entropy of a discrete 'subsystem.'

3.1 Measure-theoretic

We use the convention that $0 \ln 0 = 0$. In the following, we fix a probability space $(X, \mathcal{B}, \mu)$, and it is understood that all definitions are implicitly *with respect to the measure μ.*

Definition 16. A finite *partition* $\eta = \{A_1, \dots, A_n\}$ *of* X is a collection of disjoint subsets of X such that $\cup_i A_i = X$.

Given two partitions $\eta = \{A_i\}, \xi = \{B_j\}$, we define their *join* to be $\eta \vee \xi := \{A_i \cap B_j\}$. We denote the iterated join of a finite collection of partitions $\eta_1, \dots, \eta_k$ by $\bigvee_\ell \eta_\ell$.

Given $(X, \mathcal{B}, \mu)$ as above, the *(measure-theoretic) entropy* $h(\eta)$ of a finite partition $\eta = \{A_1, \dots, A_n\}$ is defined as

$$h(\eta) := -\sum_i \mu(A_i) \ln \mu(A_i) \ . \tag{17}$$

Remark 18. There is a one-to-one correspondence between finite partitions of X and finite σ-subalgebras of $\mathcal{B}$: To a partition η we associate the subalgebra $\mathcal{A}(\eta) \subseteq \mathcal{B}$ on all finite unions of elements of η, and to a finite σ-subalgebra $\mathcal{C} = \{C_i\} \subseteq \mathcal{B}$ we associate the partition $\eta(\mathcal{C})$ whose elements are finite intersections $C_{i_1} \cap C_{i_k} \cap C^c_{j_1} \cap C^c_{j_\ell}$.

Remark 19. One way to think about the entropy of a partition is as follows: For a measurable subset $A \subseteq X$ of a probability space, consider the quantity $-\ln \mu(A)$. We can think about this number as measuring the amount of information, or how 'surprising' the event A is. If $\mu(A) = 1$, i.e. A occurs with probability 1, then it is not surprising: $-\ln 1 = 0$. However, if $\mu(A) \ll 1$, i.e. A is a rare event, then it is very surprising when it does occur: $-\ln \mu(A) \gg 1$. The quantity $\ln \mu(A)$ is called the *Shannon information* of the event A, and the entropy is the expectation of the information.

If $T:(X,\mathcal{B},\mu)\to(Y,\mathcal{B}',\mu')$ is a measure-preserving transformation of probability spaces, then T^{-1} takes finite partitions of Y to finite partitions of X.

Note that the entropy of a partition η is equal to the entropy of any finite collection $\eta'=\{C_1,\ldots,C_n\}$ of subsets of X with the property that $\mu(A_i\setminus C_i\cup C_i\setminus A_i)=0$ for all i.

Notation 20. Let $\mathcal{C},\mathcal{D}$ be σ-subalgebras of the σ-algebra $\mathcal{B}$. We write $\mathcal{C}\overset{\circ}{\subseteq}\mathcal{D}$ if for all $C\in\mathcal{C}$, there exists $D\in\mathcal{D}$ such that $\mu(C\setminus D\cup D\setminus C)=0$, and we write $\mathcal{C}\overset{\circ}{=}\mathcal{D}$ if the previous condition also holds with $\mathcal{C},\mathcal{D}$ interchanged.

Given the above discussion, the definition of *conditional entropy* $H(\xi/\eta)$ follows naturally: it is the the expectation of (information given by the events of the partition ξ, given that we know the outcome of η):

$$H(\xi/\eta):=-\sum_{A\in\eta}\mu(A)\sum_{B\in\xi}\frac{\mu(A\cap B)}{\mu(A)}\ln\frac{\mu(A\cap B)}{\mu(A)} \tag{21}$$

It turns out that the function $-\sum_i\mu(A_i)\ln\mu(A_i)$ is universal in some sense:

Theorem 22. *[12] Let $\Delta^n:=\{(x_0,\ldots,x_n)\in\mathbb{R}^{n+1}\mid\sum_i x_i=1\quad x_i\geq 0\}$, and consider Δ^n as a subset of Δ^{n+1} via the inclusion $(x_0,\ldots,x_n)\mapsto(x_0,\ldots,x_n,0)$. Write $\Delta:=\cup_{n\geq 0}\Delta^n$.*

Suppose we are given a function $g:\Delta\to\mathbb{R}$ satisfying

1. *$g\geq 0$, and $g(x_0,\ldots,x_n)=0$ if and only if $x_i=1$ for some i.*
2. *g is continuous, i.e. its restriction to each Δ^n is continuous*
3. *g is symmetric*
4. *The restriction of g to Δ^n achieves a maximum at the point $\left(\frac{1}{n},\ldots,\frac{1}{n}\right)$*
5. *$g(\xi)=g(\eta)+g(\xi/\eta)$.*

Then there exists some constant $C>0$ such that $g(x_0,\ldots,x_n)=-C\sum_{i=0}^n x_i\cdot\ln(x_i)$ on each Δ^n.

Thus we can also think of entropy (of a partition) as measuring the *level of granularity* that the partition η sees.

Definition 23. Let $T:(X,\mathcal{B},\mu)\to(X,\mathcal{B},\mu)$ be a measure-preserving transformation of a probability space and η a finite partition of A. The *entropy of T relative to the partition* η is given by

$$h^*(T;\eta):=\lim_{N\to\infty}\frac{1}{N}h\left(\bigvee_{i=0}^{N-1}T^{-i}(\eta)\right)\ . \tag{24}$$

Note that while the definition of entropy of a partition takes a 'space average,' here we can think of taking the limit over N as a 'time average.'

That the limit in (24) exists follows from the

Proposition 25. *Let $\{a_n\}_{n\geq 1}$ be a sequence of real numbers such that $a_{n+k}\leq a_n+a_k$ for all n,k. Then $\lim_{n\to\infty}\frac{a_n}{n}$ exists and equals $\inf_n\frac{a_n}{n}$.*

Definition 26. The *(measure-theoretic) entropy of* a measure-preserving map $T:X\to X$ is given by the supremum over all finite partitions

$$h^*(T)=\sup_\eta h^*(T;\eta)\ . \tag{27}$$

Remark 28. It is immediate from the definition that the entropy of the identity map is zero, and for any T, $h^*(T)\geq 0$ (possibly ∞).

By the above discussion on entropy of a partition, we can think about entropy as measuring 'the rate at which the inverse image map T^{-1} increases granularity.'

Let T be the double cover map on S^1 If we let α be the open cover $\{(0-\varepsilon, \pi+\varepsilon), (\pi-\varepsilon, 0+\varepsilon)\}$, then $\alpha \vee T^{-1}\alpha$ has cardinality 4, and $\alpha \vee T^{-1}\alpha \vee T^{-2}\alpha$ has cardinality 8. Thus we might guess that the entropy of this map to be around $\ln 2$. In fact, we have the

Proposition 29. *[4] Let $f : S^1 \to S^1$ be a continuous self-map of the circle (not necessarily a group homomorphism or rotation.) Then $h^*(f) \geq \ln|\deg f|$, where $\deg f$ is the topological degree of f.*

Note that the previous example establishes a connection between the rate of expansion of f and its entropy. We will make the connection between expansion and entropy precise in considering the entropy of geodesic flow on the universal cover of a Riemannian manifold with negative sectional curvature.

3.1.1 Properties & calculations

Theorem 30. *Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$.*

1. *For $k > 0$, $h^*(T^k) = k \cdot h^*(T)$.*
2. *If T is invertible, then $h^*(T^k) = |k| \cdot h^*(T)$ for all $k \in \mathbb{Z}$.*

Proof of 1. We show that $h^*\left(T^k, \bigvee_{i=0}^{k-1} T^{-i}\xi\right) = k \cdot h^*(T, \xi)$:

$$\lim_{n\to\infty} \frac{1}{n} h^* \left(\bigvee_{j=0}^{n-1} T^{-kj} \left(\bigvee_{i=0}^{k-1} T^{-i}\xi \right) \right) = \lim_{n\to\infty} \frac{k}{nk} h \left(\bigvee_{j=0}^{nk-1} T^{-j}\xi \right) = k \cdot h(T, \xi) \ . \qquad \square$$

Next, we state some properties of h^* which allow us to compute our first examples.

Proposition 31. *Let T be a measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$, and let $\mathcal{A}, \mathcal{C}$ be finite σ-subalgebras of $\mathcal{B}$.*

1. *$h^*(T, \mathcal{C}) \leq h^*(T, \mathcal{A}) + h(\mathcal{C}/\mathcal{A})$*
2. *$h^*(T, \mathcal{A}) = h^*(T, T^{-1}(\mathcal{A}))$*
3. *If T is invertible, then for any $k \geq 1$, $h^*(\mathcal{A}) = h^*\left(\bigvee_{j=-n}^{n} T^n(\mathcal{A})\right)$*
4. *If $\mathcal{A}_n$ is an increasing sequence of finite σ-subalgebras of $\mathcal{B}$, and $\mathcal{C}$ is a finite σ-subalgebra of $\mathcal{B}$ such that $\mathcal{C} \mathring{\subset} \bigvee_n \mathcal{A}_n$, then $h\left(\mathcal{C}/\bigvee_n \mathcal{A}_n\right) \to 0$ as $n \to \infty$.*

Theorem 32 (Kolmogorov-Sinai)**.** *Let T be an invertible, measure-preserving transformation of a probability space $(X, \mathcal{B}, \mu)$, and let $\mathcal{A}$ be a finite σ-sub algebra of $\mathcal{B}$ such that $\bigvee_{n=-\infty}^{\infty} T^n(\mathcal{A}) \mathring{=} \mathcal{B}$. Then $h^*(T) = h(T; \mathcal{A})$.*

Proof. Let $\mathcal{C} \subseteq \mathcal{B}$ some finite σ-subalgebra. Then by the previous proposition,

$$\begin{aligned} h^*(T, \mathcal{C}) &\leq h^*\left(T; \bigvee_{j=-n}^{n} T^n(\mathcal{A})\right) + h\left(\mathcal{C}/\bigvee_{j=-n}^{n} T^n(\mathcal{A})\right) \\ &= h^*\left(T; A\right) + h\left(\mathcal{C}/\bigvee_{j=-n}^{n} T^n(\mathcal{A})\right) \ , \end{aligned}$$

and the term on the right side goes to 0 as $n \to \infty$ by Proposition 31. $\square$

Proposition 33. *The two-sided $(P, \vec{p})$ Markov shift has entropy $-\sum_{i,j} p_i p_{ji} \ln p_{ji}$.*

Proof. Consider the partition $\xi = \{A_i\}_{i=0,1,\ldots,k-1}$ where $A_i = \{(x_n)_{n\in\mathbb{Z}} \mid x_0 = i\}$. Then by definition of the σ-algebra on $\prod_{\mathbb{Z}}\{0,1,\ldots,k-1\}$, the finite σ-subalgebra $\mathcal{A}(\xi)$ associated to ξ satisfies $\bigvee_{n\in\mathbb{Z}} T^n\mathcal{A} = \mathcal{B}$. Therefore we can apply Theorem 32:

$$h^*(T) := \lim_{N\to\infty} \frac{1}{N} h\left(\bigvee_{i=0}^{N-1} T^{-i}(\xi)\right) .$$

A typical element of $\xi \vee T^{-1}\xi \vee \cdots \vee T^{N-1}\xi$ is

$$A_{i_0} \cap T^{-1}A_{i_1} \cap \cdots \cap T^{-(N-1)}A_{i_{N-1}} = \{(x_n) \mid x_0 = i_0, \ldots, x_{N-1} = i_{N-1}\}$$

which has measure $p_{i_0}p_{i_1 i_0}\cdots p_{i_{N-1}i_{N-2}}$, so

$$\begin{aligned}\frac{1}{N} h\left(\bigvee_{i=0}^{N-1} T^{-i}(\xi)\right) &= -\sum_{i_0,i_1,\ldots,i_{N-1}}^{k-1} p_{i_0}p_{i_1 i_0}\cdots p_{i_{N-1}i_{N-2}} \left(\ln p_{i_0} + \ln p_{i_1 i_0} + \cdots \ln p_{i_{N-1}i_{N-2}}\right) \\ &= -\sum_{i_0=0}^{k-1} p_{i_0}\ln p_{i_0} - (N-1)\sum_{i,j=0}^{k-1} p_i p_{ji} \ln p_{ji}\end{aligned}$$

Thus $h^*(T) = -\sum_{i,j=0}^{k-1} p_i p_{ji} \ln p_{ji}$. □

3.1.2 As a conjugacy invariant

Theorem 34. *If $T_i : (X_i, \mathcal{B}_i, \mu_i) \to (X_i, \mathcal{B}_i, \mu_i), i = 1, 2$ are conjugate, then $h^*(T_1) = h^*(T_2)$.*

Remark 35. h^* is not a complete invariant, i.e. there exist two non-conjugate systems with equal entropy.

3.2 Topological

The definition of topological entropy is analogous to that of measure-theoretic entropy, but we take open covers instead of finite partitions and consider the sets in an open cover to be equally weighted (instead of using a measure). Since we'd like to restrict ourselves to finite covers, we consider first compact subsets K of a topological space X and then take the supremum over all such K.

Definition 36. Let α, β be open covers of a topological space X. Their *join* $\alpha \vee \beta$ is the open cover

$$\alpha \vee \beta = \{A \cap B \mid A \in \alpha, B \in \beta\}$$

We say that β is a *refinement* of α if for all $B \in \beta$, there exists $A \in \alpha$ such that $B \subseteq A$, and we write $\alpha \leq \beta$.

Clearly we have that $\alpha \leq \alpha \vee \beta$.

Definition 37. Let α be an open cover of a topological space. Let $N(\alpha)$ be the cardinality of a minimal subcover of α. Then we define the *entropy* of α to be $h(\alpha) = \ln N(\alpha)$.

Remark 38. $\alpha \leq \beta$ implies that $H(\alpha) \leq H(B)$

The same lemma 25 implies the

Theorem 39. *Let α an open cover of X and $T : X \to X$ a continuous map. Then*

$$\lim_{n\to\infty} \frac{1}{n} H(\alpha \vee T^{-1}\alpha \vee \cdots \vee T^{n-1}) \tag{40}$$

exists.

Definition 41. We define the *topological entropy $h(T;\alpha)$ of T relative to the open cover* α to be the limit in the theorem above, and we define the *topological entropy of* T to be the supremum of the relative entropy over all open covers of X, i.e. $h(T) = \sup_\alpha h(T;\alpha)$.

The following proposition will allow us to bound the entropy of a system by a simpler 'quotient system.'

Proposition 42. *Let X_1, X_2 be compact spaces and let $T_i : X_i \to X_i$ be continuous maps. If $\phi : X_1 \to X_2$ is a continuous map such that the following diagram commutes*

$$\begin{array}{ccc} X_1 & \xrightarrow{T_1} & X_1 \\ \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle \phi} \\ X_2 & \xrightarrow{T_2} & X_2 \end{array} ,$$

then $h(T_1) \geq h(T_2)$.

3.2.1 Via separated/spanning sets

Throughout this section, (X, d) is a metric space and $T : X \to X$ a uniformly continuous (with respect to d) function. Implicitly, all definitions in this section are understood to be *with respect to d*. For some metric space (Y, δ), we denote the closed ball of radius $r \geq 0$ centered at $y \in Y$ by $\overline{B}_\delta(y, r) := \{z \in Y \mid \delta(y, z) \leq r\}$, and the open ball by $B_\delta(y, r)$.

For each $n \in \mathbb{N}$, we can consider a new metric:

$$d_n(x, y) = \max_{0 \leq i \leq n-1} d(T^i(x), T^i(y)). \tag{43}$$

Note that we can express the balls with respect to d_n as an intersection:

$$B_{d_n}(x, r) = \bigcap_{i=0}^{n-1} T^{-i} B(T^i(x), r) \tag{44}$$

Note that (fixing d and T), the balls B_{d_n} get (potentially) smaller, i.e. in view of the metric d_n, the points of X get (potentially) further apart as $n \to \infty$. If we think about y with $d(x, y) < \varepsilon$ as a 'good approximation' of x, we can think about d_n as asking: is y still still a good approximation to x after taking n applications of the function T into account?

We introduce two candidates for the definition of topological entropy, and give an indication for the proof that they are in fact the same.

Definition 45. Let $n \in \mathbb{N}$, $\varepsilon \geq 0$, and $K \subseteq X$ be a compact subset.

- A subset $F \subseteq K$ *(n, ε)-spans K with respect to T* if, for all $x \in K$, there exists $y \in F$ such that $d_n(x, y) \leq \varepsilon$.

  Let $r_n(\varepsilon, K)$ be the smallest cardinality of any (n, ε)-spanning set with respect to T.

- A subset $E \subseteq K$ is *(n, ε)-separated with respect to T* if, for all pairs of distinct points $x \neq y \in E$ we have that $d_n(x, y) \geq \varepsilon$.

  Let $s_n(\varepsilon, K)$ be the largest cardinality of any (n, ε)-separated set with respect to T.

- We define the *topological entropy of T with respect to K (and d)* to be

$$h'(T; K) = \lim_{\varepsilon \to 0} r(\varepsilon, K, T) \qquad h''(T; K) = \lim_{\varepsilon \to 0} s(\varepsilon, K, T) \ . \tag{46}$$

- Finally, the *topological entropy of T (w.r.t d)* is given by by taking the supremum over all compact subsets K of X:

$$h'(T) := \sup_{K \subseteq X \text{ cpct}} h(T; K) \qquad h''(T) := \sup_{K \subseteq X \text{ cpct}} h'(T; K) \ . \tag{47}$$

Remark 48. The intuition for the first definition is: If a function 'spreads points out,' then it should be harder to approximate a compact subset by a finite collection of points with respect to d_n as n grows; in particular, positive topological entropy corresponds to the case when the number of points needed to approximate a compact subset (under n iterations of T) grows exponentially in n.

One can think of the second definition as follows: If the ε-balls in the metric d_n shrink as n grows, we should be able to fit more non-overlapping ε-balls in the compact subset K. Positive topological entropy here corresponds to being able to fit exponentially more balls (with respect to d_n) as n grows.

That h' and h'' are equal follows from the following argument.

Claim. $r_n(\varepsilon, K) \leq s_n(\varepsilon, K) \leq r_n(\varepsilon/2, K)$.

Proof sketch. Let E be an (n, ε)-separated subset of K of maximal cardinality. Then by definition E is also a (n, ε)-spanning subset of K, so $r_n(\varepsilon, K) \leq s_n(\varepsilon, K)$. Now if E is as above and F is $r_n(\varepsilon, K) \leq s_n(\varepsilon/2, K)$-spanning, then for each $e \in E$ we can choose an $f \in F$ such that $d_n(e, f) \leq \frac{\varepsilon}{2}$. This is an injection $E \to F$, so $|E| \leq |F|$. □

Fact 49. If T is an isometry then $h'(T) = 0$.

3.2.2 Properties

Theorem 50. *let (X, d) be a metric space and $T : X \to X$ a map which is uniformly continuous w.r.t. d. If $K \subseteq K_1 \cup \cdots \cup K_m$ are all compact subsets of X, then we have that*

$$h'(T; K) \leq \max_i h'(T; K_i) \tag{51}$$

Proof sketch. For each n, we may bound $s_n(\varepsilon, K)$ by $m \cdot \max_i s_n(\varepsilon, K_i) = m \cdot s_n(\varepsilon, K_{i_n})$ for some $i_n \in \{1, \ldots, m\}$. By the pigeonhole principle, we can choose a sequence such that i_n is constant (i.e. $i_n = i_m$ for all n, m), and when we take the natural log and divide both sides by n and take the limit as n goes to ∞, the term $\frac{1}{n} \ln m$ goes to zero. □

The previous theorem allows us to bound the entropy of a system by a simpler 'subsystem.'

Dependence on the metric d Now we explore the dependence of $h'_d(-)$ on the metric d.

Theorem 52. *If d, δ are two metrics on a topological space which are* uniformly equivalent*, i.e. the identity map $(X, d) \to (X, \delta)$ is uniformly continuous (and vice versa), then for any uniformly continuous map $T : X \to X$ (note that if T is uniformly continuous in one metric, then it is in both), we have $h_d(T) = h_\delta(T)$.*

Note that it is not sufficient for d and δ to induce the same topology on X, as the following non-example shows.

Non-example 53. Let $X = \mathbb{R}_{>0}$ and $T : X \to X$ be the multiplication by 2 map. Then

- If d is the usual/Euclidean metric on $\mathbb{R}_{>0}$, then $h_d(T) = \ln 2$.
- Let δ be the metric on $\mathbb{R}$ with the property that the map $T^n : ([1, 2], d) \to ([2^n, 2^{n+1}], \delta)$ is an isometry for all $n \in \mathbb{Z}$. Since T is an isometry with respect to δ, $h_\delta(T) = 0$ by Fact 49.

Theorem 54. *Let (X, d) be a compact metric space and $T : X \to X$ a continuous map. Then $h'(T) = h(T)$, i.e. the aforementioned three definitions of topological entropy all agree.*

3.2.3 Computational tools & examples

We say that an open cover α of a topological space X is a *generator* for $T : X \to X$ if for every sequence $(A_n)_{n\in\mathbb{Z}}$ of elements of α, the intersection $\bigcap_{n=-\infty}^{\infty} T^{-n}A_n$ contains at most one point of X.

A map $T : X \to X$ of a compact metric space is said to be *expansive* if for all $\varepsilon > 0$ and all $x \neq y \in X$, there exists $n \in \mathbb{N}$ such that $d(T^n(x), T^n(y)) \geq \varepsilon$.

Theorem 55. *Let $T : X \to X$ be an expansive homemorphism of a compact metric space (X, d). Then if α is a generator for X, $h(T, \alpha) = h(T)$.*

This is the topological analogue of Theorem 32.

Corollary 56. *Let $X = \prod_{\mathbb{Z}}\{0, \ldots, k-1\}$ and let T be the two-sided shift in Example 2. Then $h(T) = \ln k$.*

Proof. Consider the generator $\alpha = \{A_\ell \mid \ell = 0, \ldots, k-1\}$ where $A_\ell = \{(x_n) \mid x_0 = \ell\}$. Then we apply the previous theorem to compute

$$\begin{aligned} h(T) = h(T; \alpha) &= \lim_{n\to\infty} \frac{1}{n} \ln N\left(\alpha \vee T^{-1}\alpha \vee \cdots \vee T^{n-1}\alpha\right) \\ &= \lim_{n\to\infty} \frac{1}{n} \ln k^n = \ln k \end{aligned}$$

□

With a little more work one can show that

Proposition 57. *Let A be a $k \times k$ matrix with entries in $\{0, 1\}$. The entropy of the topological Markov chain $T_A : X_A \to X_A$ (Example 3) is given by $h'(T_A) = \ln\lambda$ where λ is the largest positive eigenvalue of A.*

3.3 Variational principle

Let X be a metric space, and let $\mathcal{B}$ be the σ-algebra of Borel subsets of X, i.e. the σ-algebra generated by the open sets in X with respect to the topology induced by the metric. For any continuous map $T : X \to X$, T induces a map of σ-algebras which we denote by $\tilde{T} : \mathcal{B} \to \mathcal{B}$. Let $M(X, T) = \{\mu$ probability measure on $X \mid \tilde{T}\mu = \mu\}$, i.e. the measures which are invariant under T.

Theorem 58. *Let T be a continuous map of a compact metric space X. Then $h(T) = \sup_{\mu\in M(X,T)} h^*_\mu(X)$.*

Remark 59. Consider the 2-sided topological Markov chain $T_A : X_A \to X_A$ with transition matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$.By Proposition 57, we have that the topological entropy of T_A is given by $h(T_A) = \ln 2$. By Proposition 33, we have that the measure-theoretic entropy of $T_A = T_P$ is $h^*(T_A) = \sum_{i,j} p_{ji}p_i \ln p_{ji} = \ln 2$, so we see that the supremum in Theorem 58 is achieved.

4 Geodesic flow

For the rest of this note, we let $(M, |\cdot|)$ be a smooth Riemannian manifold[1]. (Occasionally we omit 'smooth' or 'Riemannian.') Unless otherwise stated, assume M is compact, connected, and complete.

Let $v \in T_xM$ a tangent vector at M and let $\gamma : \mathbb{R} \to M$ the unique geodesic such that $\gamma(0) = x$ and $\gamma'(0) = v$. Note that because geodesics are constant speed [13, 5.5], the parallel translate of v along γ has constant length, and the following is well-defined:

Definition 60. Given an n-dimensional Riemannian manifold M, we write SM for the *unit tangent bundle*, i.e. the collection of all tangent vectors $v \in TM$ with $|v| = 1$.

[1] While we need less than C^∞, we do not concern ourselves here with obtaining minimal hypotheses

The *geodesic flow* on a complete Riemannian manifold is the flow on the unit sphere bundle:

$$\begin{aligned}\phi_t : &\mathrm{S}M \to \mathrm{S}M \\ &v \mapsto \text{parallel translate of } v \text{ to } \gamma(t)\ .\end{aligned}$$

5 Entropy of geodesic flow

In this section, we explore how the entropy of geodesic flow has a 'geometric' interpretation by showing it is bounded from below by the growth of volume of a ball of radius r in the universal cover. We show that the bound is not sharp in §5.2.

Denote the universal cover of M by $\tilde{M}$. Note that the metric on M pulls back to a metric on $\tilde{M}$; in particular, this gives a natural volume form on $\tilde{M}$.

5.1 Entropy and volume growth

The proofs in this section follow the paper [14].

Proposition 61. *Let M be a complete, compact, connected Riemannian manifold with $\partial M = \varnothing$. For $x \in \tilde{M}$, let $B(x,r)$ be the ball of radius r and center x, and $V(x,r)$ the volume of $B(x,r)$. Then*

$$\lim_{r\to\infty} \frac{1}{r} V(x,r) =: \lambda \tag{62}$$

exists and is independent of $x \in \tilde{M}$.

Proof. Let $N \subseteq \tilde{M}$ be a fundamental domain, and let $a > 0$ be the diameter of N. Then for all $r > a$ and all $x, y \in N$, we have that

$$B(x, r-a) \subseteq B(y,r) \subseteq B(x, r+a).$$

Therefore, for all $r > a$ and all $x, y \in N$,

$$V(x, r-a) \le V(y,r) \le V(x, r+a)$$

so if the limit in (62) exists, it is unique.

Next we show that the limit exists: For any $r, s > 0$, we have that $B(x, r+s) \subseteq \bigcup_{y\in B(x,r)} B(y,s)$, so if $\mathcal{S}$ is a maximal set of points in $B(x,r)$ which are pairwise at least $\frac{b}{2}$ apart, then

$$|\mathcal{S}| \le \frac{V(x, r+ {}^{b}\!/_{2})}{\inf_{z\in B(x,r)} V(z, {}^{b}\!/_{2})} \qquad \text{and} \qquad B(x, r+s) \subseteq \bigcup_{y\in\mathcal{S}} B(y, s+b).$$

We may assume that $V(x,r)$ is unbounded, otherwise the limit clearly exists and $= 0$. (For notational ease) choose $b > 0$ such that $\inf_{z\in B(x,r)} V(z, {}^{b}\!/_{2}) = 1$. Then we have that

$$\begin{aligned}V(x, r+s) &\le |\mathcal{S}| \cdot V(y, s+b) \\ &\le V(x, r+ {}^{b}\!/_{2}) \cdot V(x, s+b+a).\end{aligned}$$

By moving each $y \in \mathcal{S}$ closer to x by ${}^{b}\!/_{2}$ and enlarging the balls accordingly, we see that

$$V(x, r+s) \le V(x,r) \cdot V(x, s + {}^{3b}\!/_{2} + a).$$

Let us fix $s > 0$, and write $A = {}^{3b}\!/_{2} + a$. Then the last inequality allows us to control $V(x,r)$ as we 'increment by s': If $ks \le r < (k+1)s$, then

$$\begin{aligned}V(x,r) &\le V(x,(k+1)s) \le V(x,ks)V(x,s+A) \\ &\le V(x,s) \cdot V(x,s+A)^k \\ \therefore \frac{1}{r}\ln V(x,r) &\le \frac{1}{r}\ln V(x,s) + \frac{k}{r}\ln V(x,s+A) \\ &\le \frac{1}{r}\ln V(x,s) + \frac{1}{s}\ln V(x,s+A).\end{aligned}$$

Letting r go to infinity, we see that $\limsup_{r\to\infty} \frac{1}{r} \ln V(x,r) \le \frac{1}{s} \ln V(x, s+A)$. Since s was arbitrary, we can now pass to the limit as $s \to \infty$ to obtain:

$$\limsup_{r\to\infty} \frac{1}{r} \ln V(x,r) \le \liminf_{s\to\infty} \frac{1}{s} \ln V(x,s+A) = \liminf_{s\to\infty} \frac{1}{s} \ln V(x,s)$$

hence the limit exists. ☐

Example 63. When M^{n+1} has constant sectional curvature $\kappa < 0$, then by [6, III.4.1],

$$V(x,r) = C \int_0^r \left(\frac{1}{\sqrt{-\kappa}} \sinh \kappa\rho \right)^{\dim M - 1} d\rho \sim C' e^{nkr} \ ,$$

so $\lambda = (\dim M - 1)\sqrt{|\kappa|}$.

Now we relate the rate of volume growth to the entropy of geodesic flow:

Theorem 64. *Let M be a compact Riemannian manifold and let ϕ_t be the geodesic flow on $\mathrm{S}M$. Then $h(\phi) \ge \lambda$, where λ is the growth rate defined in Proposition 61.*

Proof. Let $\delta > 0$ be small and consider $B(x, r + \delta/2) \setminus B(x,r)$. Given $\varepsilon > 0$, for all r sufficiently large, we have $e^{(\lambda-\varepsilon)r} \le V(x,r) \le e^{(\lambda-\varepsilon)r}$.

Claim. $V(x, r+\delta/2) - V(x,r) \ge e^{(\lambda-\varepsilon)r}$ for some sequence $r \in \{r_1, r_2, \ldots\}$ such that $r_i \to \infty$.

Suppose not, i.e. suppose that $V(x, r+\delta/2) - V(x,r) \ge e^{(\lambda-\varepsilon)r}$ for all r sufficiently large. Then we have that for some r and all $N \gg 0$,

$$\begin{aligned} V(x, r+N\delta/2) &\le N \cdot e^{(\lambda-\varepsilon)r} + V(x,r) \\ \therefore \frac{1}{r+N\delta/2} \ln V(x, r+N\delta/2) &\le \frac{1}{r+N\delta/2} \ln \left(N \cdot e^{(\lambda-\varepsilon)r} + V(x,r) \right), \end{aligned}$$

but taking $N \to \infty$ gives a contradiction.

Returning to the proof at hand: for $r \in \{r_1, r_2, \ldots\}$, choose a maximal subset Q_r of $B(x, r+\delta/2) \setminus B(x,r)$ whose points are pairwise $\ge 2\delta$ apart. Then

$$|Q_r| \ge \frac{V(x, r+\delta/2) - V(x,r)}{\sup_{z\in M} V(z, 2\delta)} \ge C \cdot e^{(\lambda-\varepsilon)r}.$$

Since each $q \in Q_r$ can be joined to x by a geodesic of length between r and $r + \delta/2$, we now show that unit tangents for the aforementioned geodesics form a $(r+\delta/2, \delta)$-separated set for the geodesic flow on $\tilde{M}$.

Denote the projection map by $\pi : \mathrm{S}\tilde{M} \to \tilde{M}$. Let $p, q \in Q_r$ and let $v, w \in \mathrm{S}M$ the respective corresponding unit tangent vectors at x. Then we can choose a metric d_1 on $\mathrm{S}M$ such that

$$\begin{aligned} 2\delta < d(p,q) &\le d(p, \pi\circ\phi_r(v)) + d(\pi\circ\phi_r(v), \pi\circ\phi_r(w)) + d(\pi\circ\phi_r(w), q) \\ &\le \delta + d(\pi\circ\phi_r(v), \pi\circ\phi_r(w)) \\ \therefore d_1(\phi_r(v), \phi_r(w)) &\ge d(\pi\circ\phi_r(v), \pi\circ\phi_r(w)) \ge \delta. \end{aligned}$$

Thus we have that

$$h(\phi) \ge \lim_{n\to\infty} \frac{1}{r_n} \ln |Q_{r_n}| \ge \lambda - \varepsilon.$$

Since ε was arbitrary, we are done. ☐

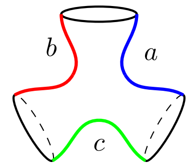

Figure 2: The pair of pants P, with segments a, b, c colored blue, red, green, respectively.

5.2 A non-example

Unlike in the case of constant sectional curvature (Example 63) where the bound obtained was sharp, we sketch an argument for producing manifolds such that the difference between λ and $h(\phi)$ is arbitrarily large, following [15]. Note that since we know the bound is sharp for compact closed manifolds of uniform sectional curvature, a non-example must involve local distortions.

Let P be the 'pair of pants' in Figure 2, and consider geodesics which remain in P for all time. Note that any such geodesic must, after $\geq \operatorname{diam}(P)$ time cross the segment a, then $\{b$ or $c\}$, and if it crosses b it must cross a or c next,... Thus the space of such geodesics maps to the product $\prod_{\mathbb{Z}}\{a, b, c\}$, and there is some finite $L > 0$ such that ϕ_L corresponds to the shift operator on the product. Thus we have shown that the space of such geodesics surjects onto the 2-sided topological Markov chain $T_A : X_A \to X_A$ with transition matrix $A = \begin{pmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{pmatrix}$. By Proposition 57, we have that the entropy of T_A is given by $h(T_A) = \ln 2$, and Proposition 42 implies that this is a lower bound for the entropy of $\phi_{\operatorname{diam}(P)}$. By our definition of entropy of a flow, $h(\phi) = h(\phi_1) = \frac{1}{|t|} h(\phi_t)$, so we can make the entropy of the geodesic flow on P arbitrarily high by shrinking P.

Adding two caps to P to obtain a disk and take the product with an $(n-2)$-dimensional ball gives a contractible n-manifold Q (with boundary) with geodesic flow of arbitrarily high entropy. Finally, we connect sum Q onto any smooth n-manifold M to increase the entropy of the geodesic flow on M while remaining diffeomorphic to the original. In sum, we have shown

Theorem 65. *[15] Any compact, smooth manifold of dimension ≥ 2 admits Riemannian metrics with arbitrarily high values of topological entropy.*

6 Ergodicity of geodesic flow

The goal of this section is to sketch the proof of the following result, modulo some technical lemmas.

Theorem 66. *[8, 7, 1, 2, 9] Let M be a compact closed Riemannian manifold which is complete and has negative sectional curvature. Then the geodesic flow $\phi_t : \mathrm{S}M \to \mathrm{S}M$ is ergodic, i.e. if $f : M \to \mathbb{R}$ is an integrable function which is invariant respect to ϕ, then f is constant a.e.*

The proof proceeds via the following steps:

1. Given $v \in \mathrm{S}M$, define the stable and unstable manifolds W^s, W^u of v. The foliation of $\mathrm{S}M$ by the orbits of the geodesic flow, taken together with W^s and W^u are a collection of transverse foliations of $\mathrm{S}M$.
2. Show that, if f is invariant with respect to the geodesic flow, then f is constant a.e. on the leaves of W^s and W^u.
3. Show that g_t satisfies an exponential decay property as one considers the leaves of W^s along a geodesic. (The same holds for W^u with the time reversed.) This will allow us to show that $W^s(v)$ satisfies a 'weak continuity property' in v.

Note: It turns out that this exponential decay property (on a manifold satisfying the same assumptions) is a sufficient condition for a flow to be ergodic, so our proof applies to this broader class of flows.

4. The previous step will allow us to apply a Fubini-style argument, i.e. to write the volume element on SM locally as (a multiple of) $dtdsdu$ where dt, ds, du are measures on the orbits of the geodesic flow, W^s, and W^u, resp. Thus we can apply 2 to conclude that f is constant a.e.

We follow the proof in the Appendix to [3] by M. Brin, interspersed with an example/special case from [16]. An exposition involving similar ideas can be found in Chapters 5-6 of [5].

6.1 Stable and unstable manifolds

Definition 67. Let M be a compact Riemannian manifold and let ϕ_t a flow on M. For $x \in M$, the *stable and unstable sets of* x are given by

$$W^s(x) = \{y \in M \mid d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to \infty\}$$
$$W^u(x) = \{y \in M \mid d(\phi_t x, \phi_t y) \to 0 \text{ as } t \to -\infty\}\ .$$

In fact these sets are *submanifolds*.

Theorem 68 (Hadamard-Perron)**.** *Let M be a smooth manifold and let ϕ_t be a C^r flow on M. Then $W^s(\cdot)$ and $W^u(\cdot)$ are a family of C^r submanifolds of M.*

A precise statement of Theorem 68 can be found in [11, Thm. 10.1.6.], and a proof can be found in [10, Thm. 6.2.8.].

Remark 69. The stable and unstable manifolds can be pictured as 'wavefronts' for the flow.

Example 70. [16] Consider a compact surface Σ with constant negative curvature. Its universal cover cover is the hyperbolic plane, i.e. Σ can be written as a quotient $\Gamma\backslash\mathbb{H}$ where $\mathbb{H}$ is the upper half-plane $\{(x, y) \in \mathbb{R}^2 \mid y > 0\}$ equipped with the metric $\frac{1}{y}(dx^2 + dy^2)$ and $\Gamma \simeq \pi_1(\Sigma)$ is a group acting totally discontinuously on $\mathbb{H}$. We consider the (un)stable sets on S$\mathbb{H}$ first.

Let $v \in T_z\mathbb{H}$ be a unit tangent vector. Then v specifies a unique geodesic $\gamma_v : \mathbb{R} \to \mathbb{H}$ with $\gamma(0) = z$ and $\gamma'(0) = v$. Let $C(z, r)$ denote the circle (in the hyperbolic metric) with center z and radius r, and consider the collection of sets $C(\gamma(t), t)$ as $t \to \infty$. (Note: Since $z \in C(\gamma_v(t), t)$ for all t, it is nonempty.) The *positive/negative horospheres* are defined to be the limits

$$S^{\pm}(v) = \lim_{t\to\pm\infty} C(\gamma(t), |t|)\ ,$$

and the stable (resp. unstable) sets of v are the inward (resp. outward) normals to the positive (resp. negative) horospheres (see Figure 70). Note: if v is a vertical tangent vector, then the horospheres $S^{\pm}(v)$ are the horizontal line through z.

Definition 71. The *horocycle flows* $h_t^{\pm}$ are defined as

$$\begin{aligned} h_t^{\pm} :& \mathrm{S}\mathbb{H} \to \mathrm{S}\mathbb{H} \\ & v \mapsto \text{translate } v \text{ clockwise (resp. counterclockwise) along } S^{\pm}(v) \text{ at unit speed}\ . \end{aligned}$$

Proposition 72. *Let $\phi = (\phi_t)$ a continuous flow on a compact manifold M which preserves a finite measure μ on X which is positive on open sets of X. Let $f : M \to \mathbb{R}$ be measurable and ϕ-invariant. Then f is constant a.e. on (un)stable submanifolds, i.e. there exists sets N_s, (resp. N_u) of measure zero such that for any $x, y \in M$ such that $x \in V^s(x)$ (resp. $V^u(x)$) and $x, y \notin N_s$ (resp. N_u), $f(x) = f(y)$.*

A heuristic/reason one might believe the proposition is true is: Suppose we assumed f was continuous (therefore uniformly continuous, since M is compact). Let x, y lie on the same stable submanifold. By uniform continuity, for all $\varepsilon > 0$, there exists $\delta > 0$ such that $d(z, w) < \delta \implies |f(z) - f(w)| < \varepsilon$. If we choose $T \gg 0$ such that for all $t \geq T$, we have $d(\phi_t x, \phi_t y) < \delta$. Then we can use the ϕ_t-invariance of f to obtain that $|f(x) - f(y)| < \varepsilon$.

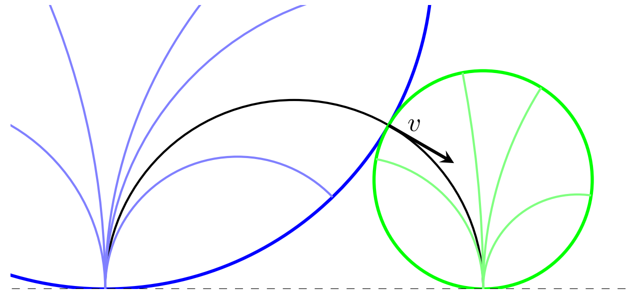

Figure 3: Positive and negative horospheres. The positive horosphere for v is given by the collection of inward normals to the thick green circle, and the negative horosphere for v is given by the collection of outward normals to the thick blue circle. The thin blue and green arcs indicate other geodesics (of vectors on the same horosphere).

While we cannot use the 'proto-argument' above, we *can* approximate f arbitrarily well by continuous functions h_m (and apply Theorem 14 to the difference $f - h_m$), which turns out to be sufficient.

6.2 Absolutely continuous foliations

Here we develop some of the theory necessary to make sense of continuity of $W^s(\cdot), W^u(\cdot)$.

Definition 73. Let N be a n-manifold. A partition V of N into connected k-dimensional $\mathcal{C}^1$-submanifolds $W(x) \ni x$ is a *k-dimensional C^0-foliation of N with C^1-leaves* if, for every $x \in N$, there is a neighborhood $U_x \ni x$ and a homeomorphism

$$\begin{aligned} w = w_x : & B^k \times B^{n-k} \to U_x \\ & (0,0) \mapsto x \qquad\qquad \text{such that} \end{aligned}$$

1. $w(B^k \times \{z\})$ is the connected component of $U_x \cap W(w(0,z))$ containing $w(0,z)$.

   Notation: given a neighborhood U of x, we will write $W_U(x)$ for $w(B^k \times \{0\}) \subseteq U \cap W(x)$.

2. $w(-,z) : B^k \to U_x \cap W(w(v,z))$ is a C^1 diffeomorphism of B^k onto $W_U(z)$ which depends continuously on $z \in B^{n-k}$ in the C^1 topology.[2]

We call the submanifolds $W(x)$ the *leaves* of the foliation W, and we say that W is a C^1 *foliation* if the w_x are diffeomorphisms.

Now we introduce two conditions on foliations which allow us to apply a Fubini-style argument, and show that one is stronger than the other

Definition 74. Let $L^{n-k} \subseteq N$ be an open local transversal for a foliation W, i.e. for all $x \in L$, $T_xM = T_xL \oplus T_xW(x)$. Let $U = \cup_{x \in L} W_U(x)$ be a union of local leaves. The foliation is *absolutely continuous* if, for any such L and U, there is a measurable family of positive measurable functions $\delta_x : W_U(x) \to \mathbb{R}$ such that for any measurable subset $A \subseteq U$,

$$m(A) = \int_L \int_{W_U(x)} \mathbb{1}_A(x,y) \delta_x(y) d\mu_{W_U(x)}(y) d\mu_L(x).$$

We call the δ_x *conditional densities.*

Definition 75. Let W be a foliation of M and consider a pair of points $x_1 \in M$ and $x_2 \in W(x_1)$. Let L_1, L_2 be two transversals to W such that $x_i \in L_i$. There are neighborhoods $U_i \ni x_i$ in L_i and a homeomorphism $p : U_1 \to U_2$, called the *Poincaré map*, such that $p(x_1) = x_2$ and $p(y) \in W(y)$ for all $y \in U_1$.

[2] A sequence of differentiable functions $(f_n)_{n\in\mathbb{N}} \subseteq C^1(M,N)$ converges to f in the C^1 topology if there exists a compact set $K \subseteq M$ such that $f|_{K^c} = f_n|_{K^c}$ for almost all n and f_n, Df_n converge uniformly to f, Df on M, TM resp.

We say that W is *transversally absolutely continuous* if p is absolutely continuous for any such transversals L_i, i.e. there exists a positive measurable function $q : U_1 \to \mathbb{R}$ such that for any measurable $A \subseteq U_1$,

$$m_{L_2}(p(A)) = \int_{L_1} \mathbb{1}_A q(x) d\mu_{L_1}(x).$$

We call q the *Jacobian* of p.

Proposition 76. *If W is transversally absolutely continuous, then it is absolutely continuous.*

Remark 77. The converse is not true in general.

Proposition 78. *Let W be an absolutely continuous foliation of a Riemannian manifold M, and let $f : M \to \mathbb{R}$ be a measurable function which is constant a.e. on the leaves of W. Then for any transversal L to W, there is a measurable subset $\tilde{L} \subseteq L$ of full induced Riemannian volume in L such that, for every $x \in \tilde{L}$, there is a subset $\widetilde{W}(x) \subseteq W(x)$ of full volume on which f is constant.*

We say that two foliations W_1, W_2 are *transversal* if, for all $x \in M$, $T_xW_1(x) \cap T_xW_2(x) = \{0\}$.

Corollary 79. *Let M be a connected Riemannian manifold, and W_1, W_2 transversal absolutely continuous foliations on M of complementary dimensions, i.e. for all $x \in M$, $T_xM = T_xW_1(x) \oplus T_xW_2(x)$ (direct sum decomposition as vector spaces, but not necessarily as inner product spaces). Assume that $f : M \to \mathbb{R}$ is measurable which is constant a.e. on the leaves of W_1 and W_2. Then f is constant a.e. on M.*

We are working with *3* foliations instead of 2, so we need to know that merging/summing the foliations (a) makes sense, and (b) preserves absolute continuity.

Definition 80. Let L_1, L_2 be transversal foliations of dimensions d_1, d_2. We say that they are *integrable* with *integral hull* W if there exists a $(d_1 + d_2)$-dimensional foliation W such that

$$W(x) = \bigcup_{y \in W_1(x)} W_2(y) = \bigcup_{z \in W_2(x)} W_1(z).$$

Lemma 81. *Let W_1, W_2 be transversal, integrable foliations of a Riemannian manifold M with integral hull W such that W_1 is $\mathcal{C}^1$ and W_2 is absolutely continuous. Then their integral hull W is absolutely continuous.*

Lemma 82. *Let W be an absolutely continuous foliation of a manifold Z and let $N \subset Z$ be a set of measure zero. Then there is a set of measure zero $N_1 \subseteq Z$ such that for any $x \in Z \setminus N_1$, the intersection $W(x) \cap N$ has conditional measure 0 in $W(x)$.*

6.3 Anosov property and Hölder continuity

We consider the special case of hyperbolic plane.

Example (70 continued). Let $\vec{\imath}$ be the unit vertical tangent vector at the point $(0, 1)$, so $\gamma_{\vec{\imath}}(t) = (0, e^t)$. We see that as the geodesic flow moves the horosphere $S^{\pm}(\vec{\imath})$ along $\gamma_{\vec{\imath}}$, the length along $S^{\pm}(\gamma_{\vec{\imath}}(t))$ is scaled by a factor of e^{-t}, and

$$g_t \circ h_s = h_{se^{-t}} \circ g_t.$$

At each point $v \in \mathrm{S}\mathbb{H}$, we have a 3-dimensional tangent space spanned by the tangent vectors to $g_t(v), h_t^+(v), h_t^-(v)$. Taking v to range over $\mathrm{S}\mathbb{H}$ gives subbundles E^o, E^s, E^u of the tangent bundle $T\mathrm{S}\mathbb{H}$; in fact they define a g_t-invariant splitting, i.e.

$$g_t(E^s_v) = E^s_{g_t(v)} \qquad \text{and} \qquad T\mathrm{S}\mathbb{H} \simeq E^o \oplus E^s \oplus E^u.$$

Furthermore, we have that $|Dg_t|_{E^o_v}| = 1$, $|Dg_t|_{E^s_v}| = e^{-t}$, and $|Dg_t|_{E^u_v}| = e^t$.

This last property motivates the following definition.

Recall. A *distribution* on a manifold M is a subspace of the tangent bundle TM which can locally be written as the span of a collection of linearly independent vector fields.

Definition 83. Given a smooth flow ψ_t on a Riemannian manifold M, we let E^o denote the tangent distribution to the orbits of the flow. A smooth flow ψ_t on a Riemannian manifold M is *Anosov, or hyperbolic* if it does not have fixed points and there are distributions $E^s, E^u \subset T_M$ and constants $C, \lambda > 0$ with $\lambda < 1$ such that for all $x \in M$ and $t \geq 0$:

1. $E^s_x \oplus E^u_x \oplus E^o_x = T_x M$.
2. $||d\phi_{t,x} v_s|| \leq C\lambda^t ||v_s||$ for any $v_s \in E^s_x$
3. $||d\phi_{-t,x} v_u|| \leq C\lambda^t ||v_u||$ for any $v_u \in E^u_x$

We refer to E^s, E^u as the stable (unstable, resp.) distributions of ϕ.

Fact 84. E^s and E^u are invariant under $d\phi_t$.

The proof of Theorem 68 shows that for any Anosov flow, the stable and unstable distributions are integrable, i.e. there exists stable and unstable foliations W^s, W^u whose tangent distributions are E^s, E^u.

Example (5 continued). The horseshoe is a hyperbolic system [5, Ch. 5]–note that it stretches the rectangle in one direction but contracts it on the other.

It turns out that Example 6.3 can be generalized to (the unit sphere bundle SM of) an arbitrary Riemannian manifold M of negative (bounded away from zero) sectional curvature: The stable (resp. unstable) subspace of a unit tangent vector $v \in \mathrm{S}M$ are the stable (resp. unstable) Jacobi fields perpendicular to the geodesic.

Proposition 85. *[3, IV.2.9] Let M be a Riemannian manifold and $\gamma : \mathbb{R} \to M$ a unit speed geodesic. Let J be a stable Jacobi field along γ perpendicular to $\dot{\gamma}$. Then if the sectional curvature of M along γ is bounded from above by $\kappa = -a^2 \leq 0$, then*

$$||J(t)|| \leq ||J(0)||e^{-at} \qquad \text{and} \qquad ||J'(t)|| \geq a||J(t)|| \quad \text{for all } t \geq 0 \ .$$

Let H_1, H_2 be subspaces of $T_x M$. We define the distance $\mathrm{dist}(H_1, H_2)$ to be the Hausdorff distance between the unit spheres in H_1 and H_2, i.e.

$$\mathrm{dist}(H_1, H_2) = \sup_{v \in H_1, |v|=1} \inf_{w \in H_2, |w|=1} d(v, w) \ .$$

We say that H_1, H_2 are *θ-transversal* if $\min_{v \in H_1 w \in H_2} ||v - w|| \geq \theta$. Writing $E^{so}_x = E^s_x \oplus E^o_x$ (and similarly for E^{uo}), we see that the compactness of M implies that there is a fixed $\theta > 0$ such that the E^s and E^{uo} are θ-transverse and E^{so} and E^{uo} are θ-transverse (independently of $x \in M$). The utility of θ-transversality lies in the

Observation 86. Let $\theta > 0$ be a real number and let $H \subseteq T_x M$ a subspace. Denote the unit sphere in a vector space V by V_1, and the projection onto a subspace $W \subseteq V$ by $(\cdot)_W$. Then for all $K \subseteq M$ which is θ-transverse, the distance function $\mathrm{dist}(-, H) : K_1 \to \mathbb{R}$ is uniformly equivalent to $\frac{|(\cdot)_K|}{|(\cdot)_H|}$, i.e. there exist constants $A, B > 0$ such that for all unit vectors $v \in K$, $A \cdot \mathrm{dist}(v, H) \leq \frac{|v_K|}{|v_H|} \leq B \cdot \mathrm{dist}(v, H)$.

Remark 87. Let H, K be the x, y-axes in $\mathbb{R}^2$. Then this is equivalent to the statement that on a compact subset $K \subseteq \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$, there always exist constants (which depend on the compact set) $A, B > 0$ such that $A \cdot \theta \leq \tan\theta \leq B \cdot \theta$ for all $\theta \in K$.

Lemma 88. *Let ψ_t be an Anosov flow. Then for every $\theta > 0$, there is some $C_1 > 0$ such that for any subspace $H \subset T_x M$ with the same dimension as E^s_x and θ-transversal to E^{uo}_x and any $t \geq 0$,*

$$\mathrm{dist}\left(d\psi_{-t,x}(H), E^s_{\psi_{-t}(x)}\right) \leq C_1 \lambda^t \mathrm{dist}(H, E^s_x) \ .$$

Proof sketch. Let $v \in H$ such that $|v| = 1$, and write $v = v_s + v_{uo}$ where $v_s \in E^s_x$ and $v_{uo} \in E^{uo}_x$. Then by definition of an Anosov flow, we have

$$|d\psi_{-t} v_s| \geq \text{const.} \cdot \lambda^{-t} |v_s| \qquad \text{and} \qquad |d\psi_{-t} v_{uo}| \leq \text{const.} \cdot |v_{uo}| \ ,$$

and the lemma follows from Observation 86. ∎

Definition 89. Let (M, d) be a Riemannian manifold. A distribution $E \subseteq TM$ is *Hölder continuous* if there are constants $A > 0, \alpha \in (0, 1]$ such that

$$\operatorname{dist}(E_x, E_y) \leq A \cdot d(x, y)^\alpha \ .$$

Remark 90. Given a hyperbolic flow ψ_t on a manifold M, it turns out we can define and *adjusted metric* which is equivalent to the Riemannian metric but interacts nicely with the flow ψ. Given $\beta \in (\lambda, 1)$, and for some $T > 0$, define $|\cdot|'$ by

$$|v_o|' = |v_o| \qquad |v_s|' = \int_0^T \frac{|d\psi_\tau(v_s)}{\beta^\tau} d\tau \qquad |v_u|' = \int_0^T \frac{|d\psi_{-\tau}(v_s)}{\beta^\tau} d\tau \ .$$

Then ψ_t is hyperbolic with respect to $|\cdot|'$ and we can take $C = 1$ if we replace λ by β in Definition 83.

Proposition 91. *The distributions E^s, E^u, E^{so}, E^{uo} of a smooth Anosov flow ψ_t are Hölder continuous.*

Note that a proof of the above statement for E^s implies a proof for E^u by reversing the time, so it suffices to show the proposition for E^s.

Proof. Write $\psi = \psi_1$. Fix $0 < \gamma < 1$, let $x, y \in M$, and choose d such that $\gamma^{d+1} < d(x, y) \leq \gamma^d$. Let $D \geq \max |d\psi|$ and fix $\varepsilon > 0$. Let m be equal to the integer part of $\frac{\ln \varepsilon - \ln \gamma^q}{\ln D} = \log_D(\varepsilon/\gamma^q)$. Then

$$d(\psi^i x, \psi^i y) \leq D^i d(x, y) \leq \gamma^q D^i \qquad \text{for } i = 0, 1, \ldots, m \ .$$

Let $\gamma \ll 1$ so that m is large. Choose a system of small coordinate neighborhoods $U_i \supset V_i \psi_i(x)$ such that $\psi^{-1}(V_i) \subseteq U_{i-1}$. Assume that ε is small enough such that $\psi_i(y) \in V_i$ for all i.

Want: to estimate the distance between E^s_x and the parallel translate of E^s_y from y to x. Let $v_y \in E^s_{\psi_m(y)}$ such that $|v_y| > 0$, and let $v_k := \psi_{-k}(v_y) := v^s_k + v^{uo}_k$. Let $w^s, w^{uo} \in E^{s,uo}_{\psi_{m-k}(x)}$ be arbitrary. Fix $\eta \in (\sqrt{\beta}, 1)$.

Claim. For $\varepsilon > 0$ sufficiently small, $\frac{|v^{uo}_k|}{|v^s_k|} < \delta \eta^k$ for some small δ.

We show this by induction: *Base case ($k = 0$):* $\frac{|v^{uo}_0|}{|v^s_0|} < \delta$ for ε sufficiently small, since $E^s_{(\cdot)}$ is continuous [5, Prop. 5.2.1.].

Inductive step: Assume the claim is true for k. Let $A_k := (d_{\psi_{m-k}(x)}\psi)^{-1}$ and $B_k := (d_{\psi_{m-k}(y)}\psi)^{-1}$. By our choice of D,

$$|A_k - B_k| \leq C \cdot \gamma^q \cdot D^{m-k} =: \xi_k \leq \text{const.} \cdot \varepsilon$$

for some constant $C > 0$ which depends on the 2nd derivatives of ψ. We also have that

$$|A_k v^s_k| \geq \beta^{-1}|v^s_k| \qquad |A_k v^{uo}_k| \leq |v^{uo}_k| \ ,$$

$$v_{k+1} = B_k v_k = A_k v_k + (B_k - A_k)v_k = A_k(v^s_k + v^{uo}_k) + (B_k - A_k)v_k \ .$$

Therefore by the triangle inequality,

$$\begin{aligned}
\frac{|v^{uo}_{k+1}|}{|v^s_{k+1}|} &\leq \frac{|A_k(v^{uo}_k)| + |B_k - A_k| \cdot |v_k|}{|A_k(v^s_k)| - |(B_k - A_k)v_k|} \\
&\leq (\delta\eta^k + \xi_k)\frac{|v_k|}{\beta^{-1}(|v_k| - |v^{uo}_k|) - \xi_k|v_k|} \\
&\leq \left((\delta\eta^k + \xi_k)\sqrt{\beta}\right)\left(\frac{1}{(1 - \beta\xi_k - \delta\eta^k)}\sqrt{\beta}\right) \ .
\end{aligned}$$

By our choice of m and D, we have that $\xi_k = \text{const.} \cdot \gamma^q D^{m-k} \leq \text{const.} \cdot \varepsilon D^{-k}$, and hence the inequality for $k+1$ for D sufficiently large, ε sufficiently small, $\sqrt{\beta} < \eta$, and δ small. For $k = m$, we get that $|v^{uo}_m|/|v^s_m| \leq \delta\eta^m$, and, by our choice of m,

$$\frac{|v^{uo}_m|}{|v^s_m|} \leq \text{const.} \cdot \gamma^{-(q+1)\frac{\ln \eta}{\ln D}} \leq \text{const.} \cdot \operatorname{dist}(x, y)^{-\frac{\ln \eta}{\ln D}} \ .$$

Finally the direct sum of two Hölder continuous distributions is Hölder continuous, so the result for E^s (resp. E^u) implies the result for E^{so} (resp. E^{uo}). □

6.4 Proof of absolute continuity and ergodicity

Proposition 92. *[3, IV.2.10] Let M be a Riemannian manifold which is complete and simply-connected with sectional curvature κ_M bounded by $-b^2 \leq \kappa_M \leq a^2$. Write d_S for the distance in the unit tangent bundle* S*M of M.*

Then for every constant $D > 0$, there exist constants $C = C(a, b) \geq 1$ and $T = T(a, b) \geq 1$ such that for all x, y with $d(x, y) \leq 1$,

$$d_S(\phi_t(v), \phi_t(w)) \leq Ce^{-at} d_S(v, w) \ , \quad 0 \leq t \leq R$$

where v and w are inward unit vectors to a geodesic sphere of radius $R \geq T$ in M with foot points x, y.

Remark 93. This result also holds for the manifolds under consideration here by applying the proposition to the universal cover.

Theorem 94. *Let M be a compact Riemannian manifold with a smooth metric of negative sectional curvature. Then the foliations W^s, W^u of* S*M into the normal bundles to the horospheres are transversally absolutely continuous with bounded Jacobians.*

Proof. Let L_1, L_2 be C^1-transversals to W^s, and let p, U_1, U_2 be as in Definition 75. Let $V_i \subseteq L_i$ be closed subsets such that V_2 contains an open neighborhood of $p(V_1)$.

Let Σ_n be the foliation of SM into 'inward' spheres of radius n (cf. Example 70), and let p_n be the Poincaré map for Σ_n and transversals L_i. Note that by construction $p_n \to p$ as $n \to \infty$.

Want to show: The Jacobians q_n of p_n are uniformly bounded in $v \in V_1$ and n.

We show this by writing

$$p_n = \phi^{-n} \circ P_0 \circ \phi^n \tag{95}$$

where $P_0 : \phi^n(L_1) \to \phi^n(L_2)$ is the Poincaré map along the vertical fibers of the natural projection $\pi :$ S$M \to M$.

Since, for $n \gg 0$, the spheres Σ_n approach the stable manifolds/horospheres W^s, the leaves of Σ_n are uniformly transverse to L_1 and L_2, and p_n is well-defined on V_1.

Let $v_i \in V_i$ such that $p_n(v_1) = v_2$. Then $\pi(\phi^n(v_1)) = \pi(\phi^n(v_2))$. Let

$$\begin{aligned} J_k^i &= \text{Jacobian of the time 1 map } \phi \text{ in the direction of } T_k^i = T_{\phi^k(v_i)} L_i \\ &= |\det(d\phi_1(\phi^k(v_i)))|_{T_k^i}| \ , \end{aligned}$$

and let J_0 denote the Jacobian of P_0. Then by (95),

$$|q_n(v_1)| \leq \left| \prod_{i=1}^{n-1} (J_k^2)^{-1} \cdot J_{(\phi^n(v_1))} \cdot \prod_{i=1}^{n-1} (J_k^1) \right| = \left| J_{(\phi^n(v_1))} \prod_{i=1}^{n-1} (J_k^1 / J_k^2) \right| \ . \tag{96}$$

By Lemma 88, for n sufficiently large the tangent plane at $\psi_n(v_1)$ to the image $\phi^n(L_1)$ is close to $E^{so}_{\phi^n(v_1)}$. Therefore, it is uniformly (in v_1) transverse to the unit sphere S$_xM$ at $x = \pi(\phi^n(v_1)) = \pi(\phi^n(v_2))$, so J_0 is uniformly bounded above in v_1.

Furthermore, we have that

$$\begin{aligned} \text{dist}\left(E^{uo}_{\phi^k(v_1)}, E^{uo}_{\phi^k(v_2)}\right) &\leq \text{const.} \cdot d(\phi_k(v_1), \phi_k(v_2))^\alpha \\ &\leq \text{const.} \cdot e^{-ak\alpha} \ , \end{aligned}$$

where the first inequality is by Hölder continuity of E^{uo} and the second is due to Proposition 92. Thus we have that

$$\mathrm{dist}(T_{\phi^k(v_1)}L_1, T_{\phi^k(v_2)}L_2) \leq \mathrm{const.} \cdot e^{-\beta k}$$

for some $\beta > 0$. Our assumptions on our metric imply that $d\phi_1(v)$ is Lipschitz in v, so therefore $|J_k^1 - J_k^2| \leq \mathrm{const.} \cdot e^{-\gamma k}$ for some $\gamma > 0$, and compactness of M implies that $|J_k^i|$ are uniformly bounded away from zero. Thus[3] we have shown that the right hand side of (96) is uniformly bounded in v and n. That the Jacobians are uniformly bounded follows from the next lemma. ☐

Lemma 97. *Let $(X, \mathcal{U}, \mu)$ and $(Y, \mathcal{B}, \nu)$ be two compact metric spaces with Borel σ-algebras and σ-additive Borel measures and let $p_n : X \to Y$, $n = 1, 2, \ldots, p : X \to Y$ be continuous maps such that*

1. *All p_n and p are homeomorphisms onto their images,*
2. *$p_n \to p$ uniformly as $n \to \infty$,*
3. *There exists a constant C such that $\nu(p_n(A)) \leq C\mu(A)$ for all $A \in \mathcal{U}$ and all n.*

Then $\nu(p(A)) \leq C\mu(A)$.

Definition 98. Let M be a compact Riemannian manifold. Writing m for the Riemannian volume form on M and λ_x for the Lesbesgue measure on $\mathrm{S}_x M$, we define the *Liouville measure* $d\mu$ on $\mathrm{S}M$ to be $d\mu(x, v) = dm(x) \times d\lambda_x(v)$.

Claim. The Liouville measure is invariant under the geodesic flow.

We now have all the tools at our disposal to prove Theorem 66.

Proof of Theorem 66. Because the foliation W^u is absolutely continuous and W^o is C^1, the integral hull W^{uo} is absolutely continuous by Lemma 81. Let f be a ϕ-invariant function. By Proposition 10, we can WLOG assume that f is strictly ϕ-invariant. By Proposition 72, there is a set of measure zero N_u such that f is constant on the leaves of W^u in $\mathrm{S}M \setminus N_u$. By Lemma 82, there is a set of measure zero N' such that N_u is a null set in each $W^{uo}(v)$ and in $W^u(v)$ for any $v \in \mathrm{S}M \setminus N'$. Thus f is constant a.e. on the leaf W^u, and, since f is strictly ϕ-invariant, it is constant a.e. on the leaf W^{uo}. Hence f is constant a.e. on the leaves of W^{uo}, and we can conclude by Corollary 79. ☐

7 Conclusion

In this note, we've barely scratched the surfaces of the study of dynamical systems and dynamical properties of geodesic flow. In particular, we fail to treat spectral methods, we do not define mixing, and we omit discussion of Poincaré's method, which inspired the name of the map in Definition 75. Furthermore, one can show that the geodesic flow on surfaces of negative curvature are Bernoulli [17] and are exponentially mixing (see, e.g. [18] and the references therein). However, we hope this exposition motivates the reader[4] to further one's study in some of the areas mentioned above.

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[3] Using the fact that an infinite product $\prod_{i=1}^{\infty}(1 + p_i)$ converges if and only if $\sum_{i=1}^{\infty} p_i$ converges.

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